<u>BLACK HOLES</u>: From Which Nothing May Escape,

By Erin O'Connor (1987, 1988)

QUANTUM MECHANICS AND THERMODYNAMICS OF BLACK HOLES

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ABSTRACT:

There are two types of black holes. Large cosmological black holes arise from gravitational collapse of supernova remnants. Primordial black holes were formed by implosive forces during the big bang. The former type are restricted to masses greater than 2-3 solar masses--the latter are not. The general laws of thermodynamics can be extended to encompass black hole physics. The entropy of a black hole is related to its event horizon surface area and the black holes temperature is given by its surface gravity. Black holes radiate a thermal black body spectrum in the form of "Hawking Radiation" by inducing an asymmetrical tidal force on "virtual" particles (time-energy uncertainty fluctuations) at the event horizon. Hawking radiation is negligible for large stellar black holes, but in theory should be very prominent in small primordial black holes. Primordial evaporation, if ever detected, should provide a valuable clue to the nature and structure of matter itself.

THERMODYNAMICS OF BLACK HOLE HEAT ENGINES

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ABSTRACT:

The general laws of thermodynamics can be extended to encompass black hole physics. The entropy of a black hole is related to its event horizon surface area and a black hole's temperature is given by its event horizon surface gravity. Thermodynamically analogous black hole Carnot cycles can thus be hypothesized. It is found though that for a Schwarzschild black hole it is impossible to construct such a Carnot cycle. This is due to the fact that the external parameters, black hole temperature and black hole entropy, can not be varied independently of each other. A Schwarzschild black hole, however, can function as a heat sink for some other external engine. Such an engine is examined in detail and its thermodynamic efficiency is discussed.

APPENDIX A: BLACK HOLE THEORY

A black hole may be thought of as any object of mass M with radius R where the escape velocity at its surface is greater than the speed of light. The radius at which the escape velocity *is* the speed of light is designated the event horizon and is later shown to equal the Schwarzschild radius R_S. Such an object appears black since due to special relativity light would be captured but could never escape--requiring infinite energy to do so. The Schwarzschild radius can be classically determined by solving for R in the escape velocity expression below and replacing the escape velocity V_{esc} with the speed of light C.

$$V_{esc} = \sqrt{\frac{2GM}{R}} \Rightarrow R = \frac{2GM}{V_{esc}^2} \Rightarrow R_s = \frac{2GM}{c^2}$$

Einstein's theory of General Relativity imposes even further restrictions. It can easily be shown that, due to General Relativistic gravitational time dilation, time essentially "stops" at the event horizon of a black hole as perceived by an observer outside the event horizon. Thus *nothing* may escape the event horizon ~~(see Appendix A)~~. In non General Relativistic black hole theory a particle may not possess enough energy to escape from within the event horizon of a black hole out to infinity, but may travel some finite distance away from the black hole (past the event horizon) before losing all its energy and plummeting back (as a rock on the Earth does when tossed up). If it were not for the general relativistic constraint, a hypothetical observer outside the Schwarzschild radius could then conceivably obtain information from within the black hole (in the form of modulated light pulses--for example). Such a process is strictly forbidden relativistically--the event horizon being an ABSOLUTE barrier from which <u>nothing</u> may escape, not even temporarily. A better interpretation of the event horizon is to consider it as a one way membrane where particles and radiation may be absorbed, but from which neither may be emitted.

Soon after Schwarzschild obtained the space-time geometry outside a spherical object of mass M, two scientists H. Reissner in 1916 and G. Nordstrom in 1918 independently solved Einstein's equations to find the space-time geometry outside a spherical object of mass M and charge Q.[24] Then in 1963 R. P. Kerr achieved the next significant step by determining the space-time geometry outside a rotating object of mass M and angular momentum S.[24] Finally in 1965 E. T. Newman and his collaborators solved the problem of space-time geometry outside an object of mass M, electric charge Q, and angular momentum S.[24] This solution describes the most general type of black hole and is commonly referred to as a Kerr-Newman black hole. As we will see, Price's "No Hair" theorem (see Appendix C) suggests that these three parameters represent the only measurable quantities in the final state of a collapsing body.[24]

APPENDIX B: QUANTUM MECHANICAL BLACK HOLE THEORY

Although general relativity has allowed a greater comprehensive understanding of black holes, there are still some important questions which have defied explanation--the most notable of which involve entropy considerations. It is reasonable to assume that certain physical quantities of black cloud of an indefinitely large number of particles of indefinitely low mass. Due to the uncertainty principal of quantum mechanics, however, the de-Broglie wavelengths of each constituent particle must be smaller than the size of that particle.[19] It thus appears that the number of configurations which could lead to a specific black hole, although very large, must be finite. The logarithm of the number would therefore be a measure of the amount of information that was irretrievably lost through the event horizon during the collapse.[19]

APPENDIX C: A BLACK HOLE HAS NO HAIR

R. H. Price, by linearizing Einstein's General Relativistic field equations, hoped to reduce their complexity enough so that, by considering a few special cases, he may understand the general nature of what happens to the physical properties of an object as it gravitationally collapses into a black hole.[24] He found that each initial irregularity in the collapsing object tends to be smoothed out, with the exception of a select few. The smoothing out process takes place through radiation. The critical property which determines which information survives and what is radiated away is the spin of the basic interaction conveying the information.[24] As the collapse proceeds, "only those details remain which are communicable to the outside observer as moments of order less than the spins of transmitting interactions."[①] The implications are that when a black hole is created by

[①] I do not quite understand what this means myself--but that's exactly what it says on pp.59-60 of Jayant Narlikar's Violent Phenomena in the Universe.

holes be conserved and that the entropy of black holes should be greater than that of its constituent elements. This is evident from the fact that, as unconventional as a black hole might be with respect to ordinary matter, it must still co-exist in a universe where classical and relativistic laws are valid.

First we must consider how to express and measure the entropy of a black hole. Many striking similarities have been found between thermodynamic entropy and the area of a black hole's event horizon. The equations of state relating a black hole's Schwarzschild surface area with the Schwarzschild surface gravity are identical in form to the thermodynamic equations relating entropy to temperature. Indeed, this analogy can be carried far enough to say that the surface area is a measure of entropy, and the surface gravity is a measure of the black holes temperature. This relationship between entropy and black hole surface area, although mathematically easily substantiated, is intuitively quite difficult to grasp.

Bekenstein suggested that one could interpret the entropy of a black hole as proportional to the logarithm of the number of configurations which upon gravitational collapse result in the same final state of a black hole (a state characterized solely by mass, angular momentum, and electronic charge).[19] Shannon's formula gives us,

$$S = -\sum_n P_n \ln P_n$$

where P_n is the probability of finding the internal configuration in the nth state.[19] Classically, this number would be infinite, since the black hole could have been formed by the collapse of a gravitational collapse, it rapidly settles down to a stationary state where only those attributes which are characterized by long range forces remain: the mass (M), the angular momentum (J), and the electronic charge (Q).[24] This state of affairs is expressed by John Wheeler in those oft-quoted words: "A black hole has no hair!"[24]

APPENDIX D: ENERGY EXTRACTION FROM A KERR-NEWMAN BLACK HOLE

PENROSE PROCESS:

A process by which energy may be liberated from a Kerr-Newman black hole is the Penrose process. It is possible for a piece of matter to enter the "ergosphere", break into two pieces, then have one of the pieces come out with more energy than both pieces had initially (see Figure 5).[24]

The ergosphere is that region outside the event horizon of a black hole which is defined as follows. Solutions to General Relativistic equations of space-time geometry indicate that if the black hole has a characteristic angular momentum, the local fabric of space-time itself will be carried along with the black holes rotation.[24] A particle brought in from infinity will have to expend energy in order to maintain a horizontally static relation against the rotational flow of space-time (see Figure 6). At the "static limit" the particle must travel at the speed of light and consume infinite energy. A particle which passes within the static limit will thus be swept away with the fabric of space-time even though it has not penetrated the event horizon. the ergosphere is not spherically symmetric. It is widest at the

The Penrose process

The process proposed by Penrose is in fact a thought experiment designed to extract energy from the rotating Kerr black hole by using the properties of the ergosphere discussed in the text. As shown in Fig. 5 the process involves dropping a mass into the ergosphere, arrange for it to split into two bits, with one bit falling inside the horizon and the other escaping outside the ergosphere.

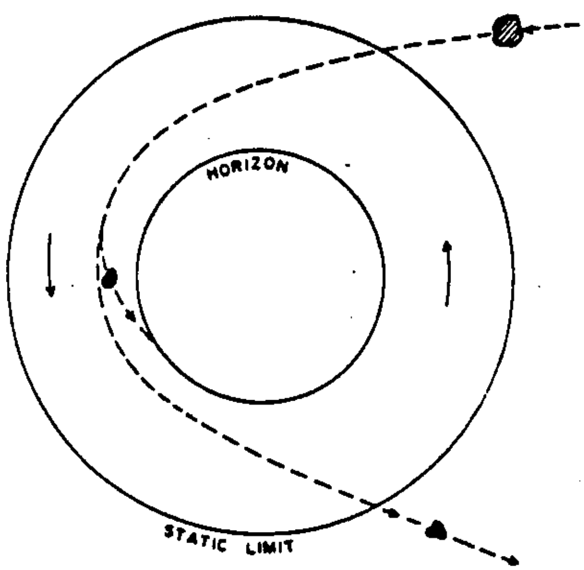

Fig. 5 In the Penrose process the energy of the incoming projectile is less than that of its part that emerges from the ergosphere. 24

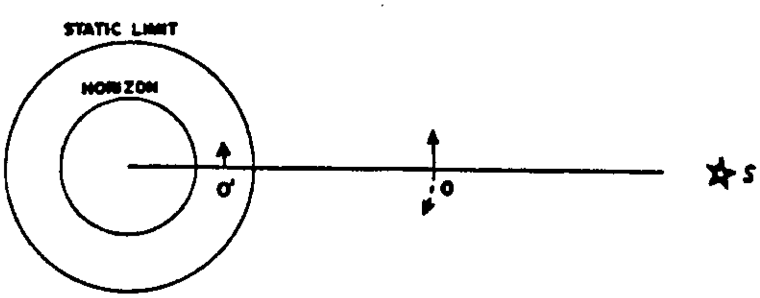

Fig. 6 The static limit is shown in its equatorial cross section. The observer O outside this limit can align himself in a fixed position relative to distant stars like S by exerting a suitable force to counter the tendency of the black hole to carry the observer in the direction of its rotation. The observer O' in the ergosphere is inexorably carried along by the rotating black hole. 24

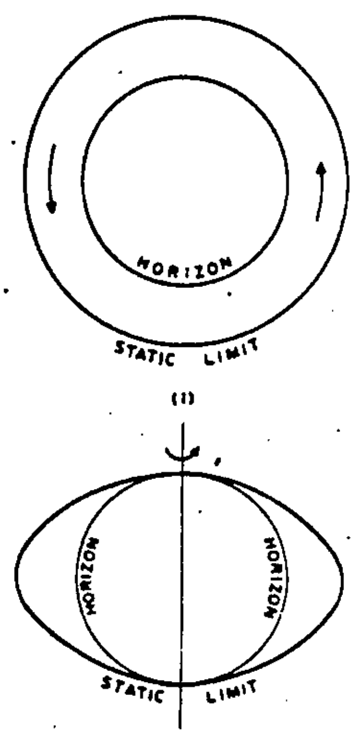

Fig. 7 The equatorial and meridianal sections of a Kerr–Newman black hole are shown respectively in (i) and (ii). The ergosphere is the region whose inner boundary is the horizon and the outer boundary the static limit. 24

equatorial plane and flattens out at the poles (see Figure 7).[24]

A hypothetical engine utilizing the Penrose process of energy extraction is depicted in Figure 8.[23] A future city, built spherically (or circularly) about a black hole, ejects its garbage into the black hole in such a way (negative energy orbits) as to generate electric power via the Penrose process. This engine is more fully discussed in Misner, Thorne, and Wheeler's book Gravitation (see reference).

The physics of this Penrose process is a simple exercise in relativistic conservation of energy where the second law of black holes must hold (this law and its implications are discussed later in this paper--in short: the surface area of a black hole may never decrease). The energy which escapes is thus not taken from the mass of the black hole (since the event horizon area may not decrease) but rather from its momentum or electric charge. Such energy liberating processes will not work if the black hole is "static"--indeed, the Penrose process is a means by which charged, high energy black holes can dissipate their energy.[24]

SUPERRADIANCE:

Superradiance is basically the Penrose Process applied to electromagnetic radiation. The effect is that of scattering incident wave packets at the same frequency but at increased amplitude--thus liberating energy from the black hole. Interestingly enough, for classical fields of half-integer spin, there is no superradiance.[17] One can understand the absence of superradiance for fermion fields as a consequence of the fact

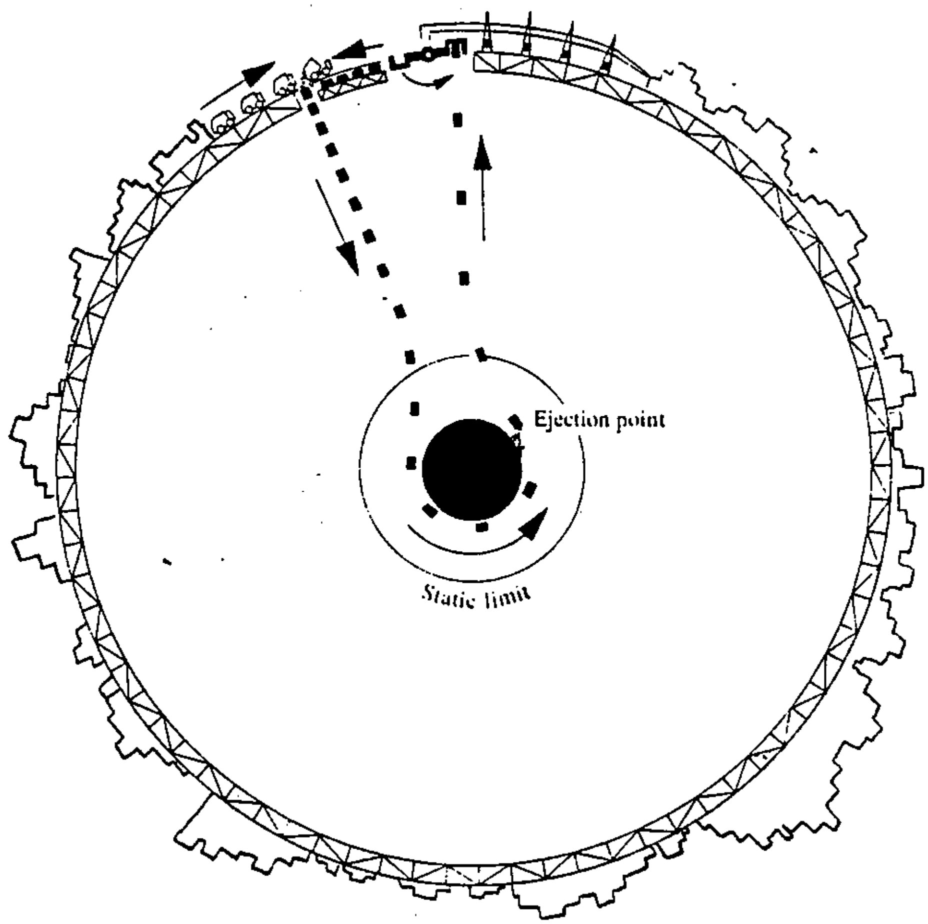

Figure [illegible]

An advanced civilization has constructed a rigid framework around a black hole, and [illegible] built a huge city on that framework. Each day trucks carry one million tons of garbage out of the city to the garbage dump. At the dump the garbage is shoveled into shuttle vehicles which are then, one after another, dropped toward the center of the black hole. Dragging of inertial frames whips each shuttle vehicle into a circling, inward-spiraling orbit near the horizon. When it reaches a certain "ejection point," the vehicle ejects its load of garbage into an orbit of negative energy-at-infinity, $E_{\text{garbage}} < 0$. As the garbage flies down the hole, changing the hole's total mass-energy by $\Delta M = E_{\text{garbage ejected}} < 0$, the shuttle vehicle recoils from the ejection and goes flying back out with more energy-at-infinity than it took down

$$\begin{aligned} E_{\text{vehicle out}} &= E_{\text{vehicle + garbage down}} - E_{\text{garbage ejected}} \\ &> E_{\text{vehicle + garbage down}}. \end{aligned}$$

The vehicle deposits its huge kinetic energy in a giant flywheel adjacent to the garbage dump; and the flywheel turns a generator, producing electricity for the city, while the shuttle vehicle goes back for another load of garbage. The total electrical energy generated with each round trip of the shuttle vehicle is

$$\begin{aligned} (\text{Energy per trip}) &= E_{\text{vehicle out}} - (\text{rest mass of vehicle}) \\ &= (E_{\text{vehicle + garbage down}} - E_{\text{garbage ejected}}) - (\text{rest mass of vehicle}) \\ &= (\text{rest mass of vehicle} + \text{rest mass of garbage} - \Delta M) - (\text{rest mass of vehicle}) \\ &= (\text{rest mass of garbage}) + (\text{amount, } -\Delta M\text{, by which hole's mass decreases}). \end{aligned}$$

Thus, not only can the inhabitants of the city use the black hole to convert the entire rest mass of their garbage into kinetic energy of the vehicle, and thence into electrical power, but they can also convert some of the mass of the black hole into electrical power! [23]

that the Exclusion Principle does not allow more than one particle in each outgoing wave packet mode and therefore does not allow the scattered wave-packet to be stronger than the incident wave packet.[17]

HAWKING RADIATION:

Hawking radiation is the black body emission spectrum radiated by a black hole at a temperature given by its surface gravity. Hawking radiation, its generation and implications, will be discussed at length in the following sections on the thermodynamics of black holes.

APPENDIX E: THE LAWS OF BLACK HOLE MECHANICS

Between 1969 and 1973, four general laws of black hole physics were developed.[24] In chronological order of "discovery" they are:

THE FIRST LAW OF BLACK HOLE PHYSICS:

This law is simply a collection of the well known conservation laws of physics which express themselves during gravitational collapse in the form of John Wheeler's "A black hole has no hair!" theorem. Of primary importance is the relativistic conservation of mass and energy

$$M' = M + m + \frac{E}{c^2}$$

which is responsible for explaining such mechanisms as the Penrose process of black hole energy extraction.

THE SECOND LAW OF BLACK HOLE PHYSICS:

This law states that in no physical process can the total event horizon surface area of all participating black holes ever decrease. The surface area A of a Relativistic Kerr-Newman black

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hole is given by[24]

$$A = 4\pi(r_+^2 + a^2) .$$

with the radius r_+ given by

$$r_+ = \frac{1}{2}\left[R_s + \sqrt{R_s^2 + 4a^2 - 4g^2}\right]$$

The non-classical additional terms result from non-Euclidean four dimensional space-time geometry. The second law thus implies that A may never decrease.[24]

THE ZEROTH LAW OF BLACK HOLE PHYSICS:

For an axisymmetric stationary black hole there exists a constant event horizon surface gravity given by

$$K = \frac{c^4}{4GM}$$

as derived by classical analysis.[24]

THE THIRD LAW OF BLACK HOLE PHYSICS:

Under no finite system of physical processes can the surface gravity of a black hole be made zero.[24]

THERMODYNAMIC RELATIONSHIP OF BLACK HOLE LAWS:

The thermodynamically suggestive taxonomy of these four basic laws of black hole physics is no coincidence. The area A is analogous to entropy, the surface gravity K is analogous to temperature, and the work done in changing the angular momentum or the electric charge of the black hole is analogous to the work done in changing the volume of a thermal system (see Table 2)[24]. By substituting these thermodynamic quantities into the above laws for black holes we arrive at the familiar four laws of thermodynamics. For example, the fourth law of black holes, that the surface gravity of a black hole can never be reduced to zero, is analogous to the fourth law of thermodynamics which states that a temperature of absolute zero can not be reached by any conceivable process in a finite time. It is this concrete analogy between black holes and thermodynamics which validates our representation of entropy as the black holes event horizon area.

It should be noted that the Hawking process makes the analogy exact in the sense that this qualitative similarity between surface gravity to temperature and of surface area to entropy is made quantitative. It is easily verified from the following section that[24]

$$\Theta \delta S = \frac{\kappa \delta A}{8\pi G} c^2$$

APPENDIX F: HAWKING RADIATION

As noted earlier, a black hole has a finite temperature proportional to its event horizon surface gravity. This would imply that a black hole could be in equilibrium with thermal radiation at some non zero temperature. The black hole should therefore emit a thermal black body spectrum of energy. Such an assumption, however, defies the very definition of a black hole.

Since energy would be radiated away, the mass of the black hole would decrease, and the second law of black holes (ever increasing surface area) violated.

The paradox was resolved in 1974 by S. W. Hawking, who was investigating the behavior of matter in the vicinity of a black hole according to Quantum Mechanics.[24] The black hole was found to emit particles and radiation as if it were an ordinary hot body with a black body temperature proportional to the surface gravity and inversely proportional to the mass.[24] This is only possible if the Second Law of Black Hole Physics is coupled with the Second Law of Thermodynamics in such a way that this new Generalized Second Law of Thermodynamics is never violated;

$$\Delta(S_{bh} + \eta A_{bh}) \geq 0$$

This law will be dealt with more rigorously later.

BLACK HOLE VIRTUAL PARTICLE EMISSION SPECTRUM:

Hawking's calculations led him to the conclusion that the number of photons of energy E per unit energy band per unit volume emitted by a black hole is given by[24]

$$N(E) = \frac{8\pi}{c^3 h^3} \frac{E^2}{e^{4\pi^2 E/\kappa h} - 1}$$

These calculations of such a quantum mechanical emission process lead directly to a black body spectrum. For a true electromagnetic black body, emitted photons are distributed according to the following spectrum[24]

$$N(E) = \frac{8\pi}{c^3 h^3} \frac{E^2}{e^{E/k\theta} - 1}$$

where N(E) denotes the numbers of photons of energy E per unit energy band per unit volume in a black body distribution of temperature θ.

These two equations are essentially equivalent if we equate the exponential terms,

$$e^{E/k\Theta} = e^{4\pi^2 E/Kh} \Rightarrow \frac{E}{k\Theta} = \frac{4\pi^2 E}{Kh}$$

It was this great similarity between the above two results which led Hawking to conclude that a black hole radiates as a black body and its event horizon surface gravity K endows it with a temperature

$$\Theta = \frac{h}{4\pi^2 k} K$$

A similar relationship holds between the event horizon surface area (A) of a black hole and its entropy[24]

$$S_{bh} = \frac{\pi k}{2hG} A$$

APPENDIX G: THE GENERALIZED SECOND LAW OF THERMODYNAMICS

Now comes the question of whether these expressions for temperature and entropy are real physical attributes of the black hole in a thermodynamic sense, or merely mathematically similar equations with no real connection to the physical concepts of temperature and entropy as we normally think of them.

The Second Law of Thermodynamics states that the total entropy of an isolated system (or the universe in general) may never decrease. S_c denotes the "common" entropy in the universe--that which we normally designate S_{tot} in the Second Law of Thermodynamics when not considering black holes.

$$\Delta S_c \geq 0$$

It seems though that classical black hole theory violates (or as Bekenstein puts it "transcends") this Second Law of

Thermodynamics.[5] Consider the accretion process. "Disorganized" matter, with a large number of possible configuration states (consider a cloud of particles--for example), has a high entropy associated with it. This is because entropy is known to be proportional to the natural logarithm of the number of possible microstates of the system.[6,27] This cloud falls into a black hole and attains an "organized" state characterized by only three configuration parameters--the mass, charge, and angular momentum (Price's "No Hair" theorem). Due to the small number of remaining configuration states--the entropy is now much lower. The entropy was thus reduced, the change in entropy being negative--thus violating the Second Law of Thermodynamics.

Alternatively, we may consider the connection between entropy and black hole accretion under a different light. The Cosmic Censorship Hypothesis points out that since the event horizon is a classically impenetrable barrier--there is no way of obtaining any information at all of what goes on within it. All known (or obtainable) information of a particle is lost when it falls within the event horizon. Since entropy can also be thought of as a measure of what is not known (or cannot be known) about a system--we see that in a sense the entropy is increased. In some way, this increase in entropy must override the decrease in entropy effect described in the paragraph preceding in order for the Second Law of Thermodynamics to remain valid for black holes. The trick now is to quantify this concept and incorporate it into, and thus extend, the applicability of the Second Law.

As matter falls into a black hole the Schwarzschild radius increases in accordance with the first and second laws of black hole physics. As the Schwarzschild radius increases, more and more "fabric of space-time" falls within the event horizon--or more accurately, the volume bounded by the event horizon. Since this volume represents a correspondingly larger and larger region of space time of which we can not know anything about--due to the Cosmic Censorship Hypothesis--and since entropy is a measure of what we can not know--the entropy of the black hole system can be thought of as increasing during the accretion process.[2] Thus we may conclude that the entropy of a black hole system is related to the volume bounded by the event horizon. Green's theorem suggests that anything of importance that occurs within a volume of interest can be measured at its surface. In this case, since we are merely interested in the volume itself, the surface to volume relationship is exceptionally simple. Thus we find that the absolute entropy of the black hole system is proportional to the surface area at the event horizon.

$$S_{bh} \propto (\text{Surface Area } A) \Rightarrow S_{bh} = \eta A$$

The Generalized Second Law of Thermodynamics (denoted as the GSL by Bekenstein) can thus be formulated as follows.[6,7]

$$\Delta S_{tot} = \Delta(S_c + S_{bh}) = \Delta(S_c + \eta A) \geq 0$$

By applying this GSL to specific thermodynamically analogous "quasi-static" processes where it is known that the change in entropy is zero, one obtains a value for the undetermined constant of $n = \frac{1}{4}h^{-1}$.[6,7] In its final form (with units such that G

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[2]The following arguments represent my own thoughts on the subject and may differ with those of Bekenstein and Hawking.

= c = k = 1), the GSL becomes,

$$\Delta(S_c + \tfrac{1}{4}\hbar^{-1}A) \geq 0 \Rightarrow \Delta S_c + \Delta(\tfrac{1}{4}\hbar^{-1}A) \geq 0$$

Since the terms in the above equation must all relate to the same physical quantities (i.e. the units for each additive term must be the same)--the area of a black hole is a direct measure of the entropy of the system and is not merely a mathematical coincidence as hypothesized earlier. One can almost go as far as to say that the event horizon area literally <u>is</u> the entropy--since this is the only reasonable external parameter from which we are able to measure and calculate the entropy.

APPENDIX H: CONSEQUENCES OF GSL--BLACK HOLE THERMODYNAMICS

Now that we have finalized the relationship between black hole physics and thermodynamics--many interesting consequences result. As determined earlier, the black hole's temperature is given by[24]

$$\Theta = \frac{\hbar}{4\pi^2 k}\kappa = \frac{\hbar c^3}{8\pi G k M} \approx 6\times10^{-8}\left(\frac{M_\odot}{M}\right)°K$$

and the black hole's entropy is[24]

$$S_{bh} = \frac{\pi k}{2hG}A \qquad \text{(which is equivalent to } \tfrac{1}{4}\hbar^{-1}A \text{ in different units)}$$

or in expanded form,[12]

$$S_{bh} = \tfrac{1}{4}M^2 - \tfrac{1}{8}Q^2 + \tfrac{1}{4}M^2\left[1 - Q^2/M^2 - J^2/M^4\right]^{1/2}$$

The thermodynamic expression dE = TdS - PdV reformulates as:[24]

$$\delta Mc^2 = \kappa\frac{\delta Ac^2}{8\pi G} + \frac{a\,\delta S_c c^2}{a^2+r_+^2} + \frac{r_+ Q\,\delta Q c^2}{a^2+r_+^2}$$

or more simply as,[12]

$$dM = TdS + \Omega dJ + \Phi dQ$$

where the temperature T, angular velocity Ω, and electric potential Φ, are given in terms of the entropy S, angular momentum J, and total charge Q, as follows,[12]

$$T = \partial M/\partial S = M^{-1}\left[1 - \frac{J^2 + \frac{1}{4}Q^4}{16S^2}\right]$$

$$\Omega = \partial M/\partial J = J/8MS$$

$$\Phi = \partial M/\partial Q = \frac{Q(Q^2+8S)}{16MS}$$

It can be seen from these thermodynamic expressions that the "heat capacity" of a black hole will be negative and destabilizing. The heat capacity is calculated from the entropy expression as follows,[12]

$$C_x = T(\partial S/\partial T)_x \Rightarrow C_{J,Q} = T(\partial S/\partial T)_{J,Q}$$

$$C_{J,Q} = \frac{8MS^3T}{J^2 + \frac{1}{4}Q^4 - 8T^2S^3} = \frac{MST}{2 - T(2M+ST)}$$

The coefficient of thermal rotation or electric charging (analogous to the coefficient of thermal expansion) is given by[12]

$$\alpha = \begin{cases} -\frac{1}{J}(\partial J/\partial T)_\Omega = -\frac{1}{J}\left[(\partial S/\partial \Omega)_T\right] = -\frac{1}{J}\left[\frac{-8M^3\Omega S}{4M^2\Omega^2+TS}\right] \\ -\frac{1}{Q}(\partial J/\partial T)_\Phi = -\frac{1}{Q}\left[(\partial S/\partial \Phi)_T\right] = -\frac{1}{Q}\left[\frac{-1}{T}\right] \end{cases}$$

The ease with which the black hole may be spun, or charged up, at constant temperature (analogous to the isothermal compressibility) is given by[12]

$$K = \begin{cases} -\frac{1}{J}(\partial J/\partial \Omega)_T \\ -\frac{1}{Q}(\partial Q/\partial \Phi)_T \end{cases}$$

these follow from[12]

$$C_\Omega - C_J = TS\alpha^2/K$$

$$C_\Phi - C_Q = TQ\alpha^2/K$$

It is also possible to deduce analogies of the Clausius-Clapeyron relation for the phase transitions (at the event horizon I assume)[12]

$$\left.\begin{array}{l} \frac{d\Omega}{dT} \\ \frac{d\Phi}{dT} \end{array}\right\} = \frac{\Delta\alpha}{\Delta K}$$

Other thermodynamic quantities and relationships can also be formulated--for example Gibb's Free Energy and the Helmholtz Function are given by[12]

$$G = M - TS - \Omega J - \Phi Q$$
$$F = M - TS$$

Maxwell's equations, of course, hold and have been employed to obtain many of the expressions given above.[12]

APPENDIX I: GEROCH PROCESS ENERGY CALCULATIONS

(To be included upon completion.)

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